needs no undue credit to make him famous—the writing alone of any one of the seven of his collected works being sufficient to rank him among the great mathematicians of his day. However, it was Gauss who in 1831, "by means of his great reputation, made the representation of imaginery quantities in the 'Gaussian plane' the common property of all mathematicians." He brought also into general use the sign i for $\sqrt{-1}$, though it was first suggested by Euler. He called $a+bi$ a *complex number* and called a^2+b^2 the *norm*.

THE POPULARIZATION OF NON-EUCLIDEAN GEOMETRY.

By GEORGE BRUCE HALSTED.

In a charming article in the *Popular Science Monthly* for January, 1901, entitled, "Geometry: Ancient and Modern," Edwin S. Crawley delightfully helps the cultured reader to get his orientation in this subject for a start into the new century.

But strangely enough this admirable paper becomes somewhat obscure when it becomes tri-dimensional. It says: "If we proceed beyond the domain of two-dimensional geometry we merge the ideas of non-Euclidean and hyper-space."

If we do so, we are apt to blunder. Just as the Bolyai plane is utterly independent of the Euclidean plane, so the triply extended space of Bolyai is utterly independent of any Euclidean space or hyper-space.

The idea that tri-dimensional Bolyai space needs four-dimensional Euclidean space is an error into which many philosophers and some mathematicians have been led, perhaps from the unfortunate adoption of the name "radius of space curvature" for the space-constant.

This blunder was refuted even before it was born by Bolyai's geodesic geometry of limit surfaces.

Thereby a Euclidean plane can be represented by a surface in Bolyai space, the theorems of Euclidean geometry find their realization as surface theorems in non-Euclidean space, where the geodesic geometry is that of Euclidean straight lines in a Euclidean plane.

Because this error about "curvature in space" is so widespread and so insidious, I treated it fully in my "Report on Progress in Non-Euclidean Geometry" to the American Association for the Advancement of Science.

The very next sentence in Professor Crawley's article reads as follows: "The ordinary triply-extended space of our experience is purely Euclidean."

Here our author states not only something which is not known, but, strangely enough, something which never can be known, which never can be proven.

Man's metric knowledge of the world independent of man, coming through imperfect instruments, for example the eye, cannot be absolute and exact.

The pure idea of a perfect plane is a creation of the human mind.

When the variations in the approximately plane surface of an actual body are minute, we deliberately make the perceived imperfections disappear, that we may identify the surface we seem to see with our ideal creation, the perfect plane. Surface is an ideal or imaginery creation to which we fit even the apparent (not real) boundaries of physical objects.

Just so the straight line is a non-real creation.

In the theoretical, the scientific, the mathematical handling of any empirical data the process is always the same. Always the results of any observations hold good only within definite limits of exactitude and under particular conditions; we replace these results with statements of absolute precision and generality.

Our replacement is only confined in its free arbitrariness in that it should seem to snuggle to the seeming facts, and must introduce no logical contradictions.

In this sense the ordinary triply-extended space of our experience is at present Euclidean or Bolyaian or Riemannian as you choose. Each is, up to the present day, in simple and perfect harmony with experience, with experiment, with the properties of the solid bodies and the motions about us.

If the angle-sum of a single rectilineal triangle is exactly a straight angle, space is Euclidean; if less than a straight angle, Bolyaian; if more, Riemannian.

The mechanics of actual bodies in the external space of our experience might conceivably be shown by merely approximate measurements (the only kind that ever were) to be non-Euclidean; just as a body might be shown to weigh more than two grams or less than two grams, though it never can be shown to weigh precisely, absolutely two grams.

In this sense the Euclidean geometry is positively hopeless, in that it never can be proven and no respectable person would now for a moment attempt to establish it, while there was nothing theoretically absurd in the claim of C. S. Peirce to be able to show that space is Bolyaian, which claim has found its way into Boyer's *Histoire des Mathématiques*, page 247, though the index confuses C. S. Peirce with his father, Benjamin Peirce.

When Lobachevski exerted himself to obtain with exceeding great precision the sum of the three angles in the very largest triangles attainable for his measurement, he found this sum did not differ from two right angles by the hundredth part of a second. This shows that the space of experience approaches the ideal Euclidean space with an approximation which is very far-reaching. But it would be strange if any educated person should need to be told that all our measurements are approximate only, and that with the approximate we can never reach the absolutely exact.

The mistake of supposing that they *know* our space, "the space in which we really live," is not Boylyaian, made by Phillips and Fisher, Professors in

Yale, in their Elements of Geometry, page 23, and even by so good a mathematician as H. Schubert of Hamburg, I have exposed in my paper, "Non-Euclidean Geometry," AMERICAN MATHEMATICAL MONTHLY, Vol. 7, pages 123-133.

A striking testimony that the non-Euclidean geometry has won its fight and henceforth must be reckoned with is found in the History of Mathematics by Boyer, Paris, 1900. He says, pages 240-7: "The last quarter of the nineteenth century has seen built up interesting theories. But beyond contradiction the most original researches of this period pertain to the *non-Euclidean Geometries*, and it is with them that we will terminate this exposé of contemporary science."

I only wish the short account which follows this prelude and terminates the book were as accurate as it is impressive and stimulating. A full-page portrait of Lobachevski is given. But of the two entirely different likenesses we possess of the great Russian, this is the conventional one of Lobachevski depressed, baffled, about to become blind and die with his great achievement unrecognized. The other portrait, a Daguerreotype from life, which I first saw at Kazan, and of which now you may see a copy as frontispiece of Engel's magnificent "Nikolaj Iwanowitsch Lobatschefskij," is a picture of Lobachevski the fighter, the dare-devil, the irrepressible, who startled and scandalized the despotic authorities of Kazan and the University by shooting off his rocket, who contemptuously overthrew the great Legendre, of Lobachevski who knew he was right against a scornful world, who has given to us a new freedom to explain and understand our universe and ourselves.

Boyer goes on as follows: "From far in the past men have striven to demonstrate the famous axiom postulated twenty centuries ago by Euclid, to-wit: Through a point can be drawn only one parallel to a given straight. These attempts remained unfruitful. However, at the end of the eighteenth century, an Italian jesuit, Saccheri, attempted to found a geometry resting on a principle different from the celebrated postulate." This is a mistake. Saccheri's was simply one more attempt to prove the postulate, and he thought he had proven it. The title of his book is: *Euclides ab omni naevo vindicatus.* THE AMERICAN MATHEMATICAL MONTHLY has the honor of being the first to publish a translation of this now famous work into any modern language.

Boyer continues: "Finally at the beginning of the nineteenth century, a Russian, Lobachevski, and a Hungarian, John Bolyai, perceived at about the same time the impossibility of this demonstration. Their works published independently one of the other had without doubt been inspired by the doctrines of the philosopher Kant, who, in a passage of his *Kritik* of pure reason, indicated a new consideration of space. For this latter, space existed *a priori*, preceding all experience, as form completely subjective of our intuition."

The statement that the work of John Bolyai was inspired by Kant is a complete mistake. It is a gratuitous assumption cut out of whole cloth. There is nothing to show that John Bolyai ever even heard of the existence of Kant. At Maros-Vásárhely I examined the papers, the correspondence, the

Nachlass of John Bolyai. There was not the slightest mention of Kant, not the remotest reference to Kant.

Lobachevski knew of Kant through the professor of physics at Kazan, Bronner, once an admirer of the *Kritik der reinen Vernunft.* But Lobachevski tells us more than once the inspiration and mental ancestry of his achievement, and there is no place for Kant.

Kant, supposing that we knew Euclid's geometry and Aristotle's logic to be true absolutely and necessarily, accounted for the paradox by teaching that this seemingly universal synthetic knowledge was in reality particular, being part of the apparatus of the human mind itself. When Boole and De Morgan made new kinds of logic of which the Aristotelian is only a special case, when Lobachevski, Bolyai and Riemann made new kinds of geometry of which the Euclidean is only a special case, then the very foundations were cut away from under the Kantian system of philosophy, and it was left like a man trying to lift himself by his own boots.

The Scotch philosophy accounts for this supposed absolute, metrically exact knowledge, by teaching that there are in the human mind certain synthetic theorems, called by them intuitions, directly God-given. Dr. McCosh summed up this Scotch doctrine in a big book entitled, "The Intuitions of the Mind, Inductively Investigated."

One of these divinely implanted intuitions was Euclid's famous parallel-postulate!! *Voila!*

"Yet," to quote a sentence from Wenley's criticism in Science of McCosh's disciple Ormond, "we may well doubt whether a thinker standing with one foot firmly planted on the Rock of Ages, and the other pointing heavenward, has struck the attitude most conducive to progress."

After a brief biography of Lobachevski and a quotation from his Introduction to New Principles of Geometry, which Introduction I have translated into English and published as volume V of the Neomonic Series, Boyer continues: "This postulated, behold how he proceeds to the development of his doctrine. He announces at the beginning the following axiom: Through a point can be drawn *many parallels* to a given straight." To say the least this is likely to produce misconception.

Lobachevski assumes that through a given point outside a given straight can be drawn many straights which never meet the given straight, but of this sheaf of non-meeters only two are parallel to the given straight, namely, the two which approach the given straight asymptotically.

Again Boyer says, page 245: "He became professor, and when he died in 1856 he occupied the position of Rector of the University where he had entered as simple student." This is another mistake. When Lobachevski died in 1856 he had been displaced from his position as Rector for ten years.

Passing on to the Riemannian geometry, Boyer says: "To construct this Geometry, its inventor rejects the postulatum and the first axiom of Euclid: two points determine a straight." In fact neither of these assumptions of Euclid

need be rejected to get a Riemannian geometry, the "Single Elliptic Geometry."

Boyer has perhaps been misled by his own paraphrases for what really occurs in Euclid. The real postulatum is as follows: "And if a straight cutting two straights makes with them angles interior and lieing on the same side, which together are less than two right angles, then the two straights indefinitely produced cut each other on the side on which these angles lie."

Needless to say, this remains true in both single and double elliptic geometry. The postulate currently taken in place of the unwieldy postulatum of Euclid is "that two straight lines, which cut one another, cannot be both parallel to the same straight line," which is credited by Playfair to Ludlam, though attributed even by Cajori to Playfair, and currently called Playfair's axiom. Even this does not help toward Riemannian spaces, for in them there are no parallels in the sense of coplanar non-meeting straights.

Finally Boyer gives us the well-known confusion in connection with Beltrami's pseudo-sphere.

John Bolyai found a surface in Lobachevskian space whose geodesic geometry is that of Euclid's straights. No one supposed that this reduced Euclidean to be only a branch of Bolyaian geometry.

Beltrami found a surface in Euclidean space whose geodesic geometry is that of Bolyai's straights. Boyer says, page 246, this reduced the Geometry of Lobachevski to be only a branch of ordinary geometry. On the contrary, the truth is that Euclidean geometry is only that special case of Bolyaian geometry made by assuming the space-constant as infinite.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

134. Proposed by **ALOIS F. KOVARIK**, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

A certain piece of land is surrounded by a four-board fence, the boards being 16 feet long. The number of acres in the land equals the number of boards in the fence. How many acres in the land?

I. Solution by **G. B. M. ZERR**, A. M., Ph. D., The Temple College, Philadelphia, Pa.; **P. S. BERG**, B. Sc., Larimore, N. D.; and **MARTIN SPINKS**, Wilmington, O.

(1) *When the field is in the form of a square.*

Let $ABCD$ be the square, O its center, $FG=l=16$ feet=the length of a panel of the fence, and $n=4$=the number of boards in a panel. Then the area of the triangle $OFG=\frac{1}{2}OE\times FG=\frac{1}{2}OE.l=n$ acres.